Frank David Faulkner

OPTIMUM SUBMARINE ROUTING.

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by

F. D. Faulkner

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UNITED STATES NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral E. J. O'Donnell, USN
Superintendent

Dr. R. F. Rinehart
Academic Dean

ABSTRACT

A numerical routine is developed to determine on a digital computer a route that will enable a submarine to go from one specified point to another in a specified time, with minimum probability of being detected, when the likelihood of being detected is a known function of the position, heading, speed, depth, and time. Special routines are required to handle bounds on the control variable and corners.

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Prepared by: F. D. Faulkner

Approved by:

J. R. Borsting
Chairman, Department of
Operations Analysis

Released by:

C. E. Menneken
Dean of
Research Administration

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OPTIMUM SUBMARINE ROUTING

Introduction and Summary

The purpose of this paper is to develop numerical routines to determine a route for a submarine such that the probability of its being detected is minimized. The basic theory is developed and a numerical routine is available for some of the simpler problems.

A typical problem is the following. The submarine is at a point x_0, y_0, which we may take to be the longitude and latitude, and it wishes to go to another point x_T, y_T, to be there at a specified time T. The probability of being detected in a time interval of length Δt is $f(x,y,u,v,w,t)\Delta t$, assuming that it has not been detected previously; u is the bearing angle of the submarine, v is its speed, w is its depth, and t is the time; f is a function describing the capabilities of the hostile observing devices. The submarine wishes to take a route such that the probability of being detected en route is minimized.

The problem is an extension of earlier work on ship routing. This problem is more difficult than those considered earlier in several ways. First, it has more variables. The computational time increases roughly as the square of the order of the system, and the numerical routines are less stable as the number of variables increases. Second, the problem has a control variable which is bounded. The depth is bounded by the maximum depth

which the submarine can stand or can go at any position, whether it is because of pressure or the danger of running aground. It is also bounded by zero, at the surface, but the likelihood of detection usually increases there and automatically eliminates any difficulty. Third, most of the data is empirical and must be fitted so as to have continuous second derivatives with respect to the variables u,v,w. Indeed the collection and fitting of the data looks like the biggest single remaining difficulty. Fourth, the curves may have corners; that is, there may be points where the submarine may need to change its heading, depth, or speed by a finite amount.

The decision was made to use the method of variation of extremals used in earlier ship-routing problems. It has the advantage of flexibility over the method of steepest ascent or gradients, which is ill adapted to problems wherein the controls are discontinuous or bounded. It was originally planned to make the routine very general so that it would handle all problems that were likely to occur. It is not clear now that this is feasible.

The problem is simplified if the coordinate set is chosen so that the path lies near some great circle, say the great circle connecting the initial and final points. Change coordinates so that this circle becomes the equator. Spherical coordinates, such as longitude and latitude, have a singularity at the poles, where longitude is undefined. If the route does not vary from this great circle by more than about 400 miles, for a route of about 6,000 miles, maximum, the flat-earth assumption is satisfactory and simplifies the computations. This depends on the fact that $\cos\theta$ is very near one whenever

$|\theta| < .1$ radian.

The theory for this problem is not readily available, if available at all, and it was necessary to include a discussion of the cases where the control lies on the boundary of the region of allowed values and where corners occur. A separate report is being written to give the underlying mathematics.

Other problems of a new type were encountered and studied. If the listening device is accurately located and if the water conditions are know accurately also, it may be possible to determine blind (deaf?) areas, regions wherein the submarine is not likely to be detected, though the probability of detection is large at neighboring or surrounding areas. The boundaries to these regions may be sharply defined and the corresponding probability generating function f is then discontinuous in the position variables. Some time was spent in trying to develop the theory for this, with limited success; with this exception, the mathematical theory is reasonably complete.

1. Statement of a Typical Problem

Let us consider for the present the following problem. The submarine is to go from one given point x_0, y_0 to another given point x_T, y_T by a given time T. The detecting or listening devices have capabilities which are known approximately; we have some knowledge of their distribution and we have made measurements with similar devices to obtain estimates of their performance. We have some data on the weather and have estimated the water characteristics. These, together with studies which we have made on similar submarines give us the conditional probability of detection. If the submarine has not been detected previously, the probability of detection in a time Δt is approximately $f(x,y,u,v,w,t)\Delta t$, and hence the probability p of being detected along the route satisfies the equation

$$dp = (1-p)f\ dt.$$

The function f is the best estimate of the detection capabilities of the hostile observers, as made up by engineers and intelligence, fitted by mathematicians and programmers. For the present, no distinction is made here between detection and classificiation.

The above equation is simplified in form if we let

(1.1) $$z = -\ln(1-p);$$

it becomes

(1.2) $$\dot{z} = f.$$

We are interested in long routes and it is assumed that the time required to change depth or speed is small compared to the total time and may be ignored.

2. Equations of Motion

Let us take the great circle containing the beginning and end points. We shall treat this like the equator, measuring longitude x in radians along it and the latitude y in radians normal to it. The third coordinate, the depth, will be denoted by w. The equations governing the system may be written then

$$\dot{x} = \frac{v \cos u}{R \cos y}$$

$$(2.1) \qquad \dot{y} = \frac{v \sin u}{R}$$

$$\dot{z} = f(x,y,u,v,w,t),$$

where R is the radius of the earth. The variables x,y,z are called the dependent or the state variables; u,v,w are called control variables.

The general problem is that of determining the control variables as functions of time to effect the desired optimization. That is, we want to pick the heading, the speed, and the depth so that the submarine will get from the specified initial point P_0 to the specified terminal point P_T with $z(T)$ a minimum. The probability of being detected along the route is, by (1.1)

$$p(T) = 1 - \exp[-z(T)],$$

and we want to choose the route to minimize this.

3. Variational Equations, Adjoint Variables.

Let us consider any route and a neighboring route. We will generate the neighboring route by replacing u,v,w on the original route by $u+\delta u, v+\delta v, w+\delta w$, and this will generate variations (first-order changes) in x,y,z satisfying the differential equations

$$(3.1)\qquad \begin{cases} \delta\dot{x} = -\dfrac{v\cos u \sin y}{R\cos^2 y}\,\delta y - \dfrac{v\sin u}{R\cos y}\,\delta u + \dfrac{\cos u}{R\cos y}\,\delta v \\ \delta\dot{y} = \dfrac{v}{R}\cos u\,\delta u + \dfrac{1}{R}\sin u\,\delta v \\ \delta\dot{z} = f_x\delta x + f_y\delta y + f_u\delta u + f_v\delta v + f_w\delta w\ ; \end{cases}$$

subscripts indicate partial derivatives, $f_x = \partial f/\partial x$, etc.

Let us introduce three Lagrange multipliers, p,q,r which are unspecified functions of t so far. We multiply each of equations (3.1) by one of these, add and integrate from 0 to T. We may write the result as

$$(3.2)\qquad \int_0^T\Big[p\Big(\delta\dot{x} - \frac{v\cos u\sin y}{R\cos^2 y}\,\delta y\Big) + q\,\delta\dot{y} + r(\delta\dot{z} - f_x\delta x - f_y\delta y)\Big]dt$$

$$= \int_0^T\Big[\Big(-p\frac{v\sin u}{R\cos y} + q\frac{v\cos u}{R} + rf_u\Big)\delta u + \Big(p\frac{\cos u}{R\cos y} + q\,\frac{\sin u}{R} + rf_v\Big)\delta v + r\,f_w\delta w\Big]dt.$$

The terms containing the variations of the dependent variables are on the left; those containing the variations of the control variables are on the right.

Now, on the left, let us integrate by parts those terms which involve the derivatives of the dependent variables, so as to eliminate their derivatives from the integral. We get, for the left side,

$$(3.3)\quad [p\delta x + q\delta y + r\delta z]_0^T - \int_0^T [\delta x(\dot p + rf_x) + \delta y(\dot q + p\frac{v \cos u \sin y}{R \cos^2 y} + rf_y) + \delta z \dot r]\, dt.$$

Now, to simplify this, <u>let us choose</u> p,q,r as solutions to the differential equations

$$(3.4)\quad \begin{cases} \dot p + rf_x = 0 \\ \dot q + p\frac{v \cos u \sin y}{R \cos^2 y} + rf_y = 0 \\ \dot r = 0. \end{cases}$$

Equations (3.4) are called the adjoint of the variational equations (3.1). When Eqs. (3.3) and (3.4 are used, (3.2) reduces to

$$(3.5)\quad [p\delta x + q\delta y + r\delta z]_0^T = \int_0^T [(-p\frac{v \sin u}{R \cos y} + q\frac{v \cos u}{R} + rf_u)\delta u + (p\frac{\cos u}{R \cos y} + q\frac{\sin u}{R} + rf_v)\delta v + rf_w \delta w]dt.$$

This is the important formula that gives us a relation between the end values of $\delta x, \delta y, \delta z$ and an integral involving the variations of the control variables $\delta u, \delta v, \delta w$. It is important that we do not have to bother with the interim values of $\delta x, \delta y, \delta z$.

It is often convenient to consider the above equations as vectorial. Let $\vec X = xi + yj + zk$, $\vec P = pi + qj + rk$, and

$$(3.6)\quad \vec F = \frac{v \cos u}{R \cos y} i + \frac{v \sin u}{R} j + fk.$$

Eq. (3.5) then becomes

$$(3.7)\quad [\vec P \cdot \delta \vec X]_0^T = \int_0^T \vec P \cdot (\vec F_u \delta u + \vec F_v \delta v + \vec F_w \delta w)dt.$$

Comments: In the above derivation, we have assumed that there are no corners. A corner is a point t_1 where any of the control variables are discontinuous. If there is a corner at t_1, and if we allow t_1 to vary, then a term must be added on the right

$$(3.8)\quad -[\vec P \cdot \vec F]_{t_1-}^{t_1+}\, dt_1;$$

[illegible] (3.9)

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible] (3.10)

[illegible]

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there is one such term for each corner.

It is also assumed that f has continuous derivatives for the moment. This is not the case in some cases of interest, but it seems simpler to discuss this extension afterward.

Index notation. We will use the vector notation given earlier whenever vector properties are to be stressed. It is also convenient to use index notation, as follows. Let x,y,z be denoted by x^1,x^2,x^3, or x^i, p,q,r by p_1,p_2,p_3, or p_i, $\frac{v \cos u}{R \cos y}$, $\frac{v \sin u}{R}$, f by f^1,f^2,f^3, or f^i, and u,v,w by u^1,u^2,u^3 or u^σ. The use of superscripts, rather than subscripts is common, particularly in differential geometry. In general, the state variables and the Lagrange mutlipliers will have Latin indices and the control variables will have Greek.

We will indicate partial derivatives by indices

$$f^i_j = \partial f^i/\partial x^j, \quad f^i_\sigma = \partial f^i/\partial u^\sigma , \tag{3.9}$$

etc. In this index notation, equations (2.1), (3.1), (3.4), (3.5) may be rewritten

$$\dot{x}^i = f^i \tag{3.10}$$

$$\delta\dot{x}^i = \Sigma f^i_j \delta x^j + \Sigma f^i_\sigma \delta u^\sigma \tag{3.11}$$

$$\dot{p}_i = - \Sigma f^j_i p_j \tag{3.12}$$

$$[\Sigma p_i \delta x^i]_0^T = \int_0^T \sum_{i\sigma} p_i f^i_\sigma \delta u^\sigma \, dt \tag{3.13}$$

It is understood that the range of both Latin and Greek indices will be 1,2,3, and if no confusion seems likely, the index of summation will not be written. This notation simplifies the writing of equations, but no notation eliminates the actual details of substuting the actual expressions into the various equations, carrying out the differentiation, etc.

Particular solutions to the adjoint. We may pick particular solutions to the adjoint so that we get the effects on various state variables of changing the control variable; G. A. Bliss did essentially this in Ballistics at Aberdeen Proving Ground during World War I. If we pick a particular solution $\vec{P}^1$ to the adjoint with the property that $\vec{P}^1(T) = 1$, then equation (3.5) reduces to

$$(3.14) \qquad \delta x(T) = (\vec{P}^1 \cdot \delta\vec{X})_0 + \int_0^T \Sigma \vec{P}^1 \cdot \vec{F}_\sigma \, \delta u^\sigma \, dt;$$

that is, this particular solution gives us the first-order effect on $\overset{=\alpha'(T)}{x(T)}$ of changing the control variables. Similarly, if the components of $\vec{P}^i$ are such that

$$(3.15) \qquad p^i_j(T) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

then

$$(3.16) \qquad \delta x^i(T) = \int_0^T \Sigma \vec{P}^i \cdot \vec{F}_\sigma \, \delta u^\sigma \, dt + (\vec{P}^i \cdot \delta\vec{X})_0 \, .$$

Note that the subscript on the p's indicates the component and the superscript indicates which state variable it is associated with.

Finally, if we have a function $g(x,y,z)_T$, to be evaluated at the endpoint of the route, we can find its variation as follows. Let us pick a particular solution $\vec{P}_g$ to the adjoint such that at time T,

$$(3.17) \qquad (\vec{P}_g)_{t=T} = \nabla g \, ,$$

gradient g being evaluated at the endpoint of the route. For, then,

$$(3.18) \qquad \delta g = \Sigma \frac{\partial g}{\partial x^j} \delta x^j$$

$$= (\vec{P}_g \cdot \delta\vec{X})_T \qquad \text{(by choice of } P_g)$$

$$= (\vec{P}_g \cdot \delta\vec{X})_0 + \int_0^T \Sigma\vec{P}_g \cdot \Sigma\vec{F}_\sigma \, \delta u^\sigma \, dt \qquad \text{(by (3.7))}$$

We will usually have problems wherein the initial point X_0 is given so that $X_0 = 0$ and the first term in the last expression will drop out. On the other hand, we will consider some problems where the terminal time T is not specified. In this case another term must be added; we will take this up when the time arises.

4. Condition for an Extremum

In this section the first necessary conditions for an extremum will be given; they will be established later.

We require of any route, to be admissible, that $x(0) = 0$, $y(0) = 0$, $x(T) = x_T$, $y(T) = y_T$, and we take $z(0) = 0$. The differential equations

$$(2.1) \qquad \begin{cases} \dot{x} = \dfrac{v \cos u}{R \cos y} \\ \dot{y} = \dfrac{v \sin u}{R} \\ \dot{z} = f \end{cases}$$

must be satisfied. In addition, the depth w must satisfy the condition

$$(4.1) \qquad 0 \leqq w \leqq W_{max}$$

where W_{max} is the maximum depth the submarine can run. In addition, the speed v is also limited,

$$(4.2) \qquad 0 \leqq v \leqq V_{max} ,$$

with V_{max} a constant of the submarine, or perhaps, a function of the depth. We are assuming that the additional noise generated whenever $v = V_{max}$ rules out $v = V_{max}$, but there may be problems where the optimum solution will require that $v = V_{max}$; so that it should go at maximum speed, the reduced time more than offsetting the increase in noise.

A route which satisfies the constraints above will be called admissible. A set of control variables which are piecewise continuous and which satisfy the constraints (4.1) and (4.2) are called allowable. Allowability is a local constraint or condition on the control variable.

We seek, among all admissible routes, the one such that $z(T)$ = minimum. Usually there is just one such route.

Minimum Principle, Euler Equations. It is a very general principle that the optimum route must satisfy the condition that the control variables must be chosen so that they maximize

(4.3) $H = \vec{P}\cdot\vec{F}$

compared with all allowable controls, for all t, $0 < t < T$, for some solution $\vec{P}$ to the adjoint. (This, with a change of sign, is the Pontryagin maximum principle.) H is called the Hamiltonian. The difficulty is that in $\vec{P}$ are three constants of integration associated with the adjoint. We may write

(4.4) $$\vec{P} = c_1\vec{P}^1 + c_2\vec{P}^2 + c_3\vec{P}^3 ,$$

the $\vec{P}$'s being defined earlier (see (3.15)).

We can choose one relation among the c's; the most convenient is to choose c_3 to be 1 (one). Then the other two must be chosen so that $x(T), y(T)$ assumed the desired values x_T, y_T.

If the values of u, v, w which minimize H lie interior to the domain of allowed values, then the Euler equations are a necessary condition

(4.5) $$\partial H/\partial u^\sigma = 0 = \vec{P}\cdot\partial\vec{F}/\partial u^\sigma$$

or, explicitly,

(4.6) $$\begin{cases} p \dfrac{v \sin u}{R \cos y} + q \dfrac{v \cos u}{R} + rf_u = 0 \\ p \dfrac{\cos u}{R \cos y} + q \dfrac{\sin u}{R} + rf_v = 0 \\ f_w = 0. \end{cases}$$

where $p = c_1 p^1 + c_2 p^2 + p^3$, $q = c_1 q^1 + c_2 q^2 + q^3$, $r = c_1 r^1 + c_2 r^2 + r^3$.

The Euler equations and the adjoint system together are here called the Euler-Lagrange equations.

It may be that H is minimized, not by an interior value of w but by w on the boundary of the allowed values,

(4.7) $$w = W_{max}$$

for part or all of the route. This is probably the case whenever the search effort is mostly surface and aerial. In this case, the last of equations (4.6) is replaced by (4.7).

5. The Numerical Routine

In this section the routine for determining the route is discussed.

Let us guess a set of values for c_1, c_2. (This choice may pose a little problem but has not in any problems run so far). Let us run the route, determining the control variables for each value of t by the minimum principle.

Let us now consider the effect on the control variables of changing c_1, c_2 by a small amount. What change would this effect in the control variables, assuming we are at any point of the route? The Euler equations (4.5) must hold on the original and the varied path, and hence

$$(5.1) \quad \sum_i \frac{\partial^2 H}{\partial u^\sigma \partial c_i} dc_i + \sum_\tau \frac{\partial^2 H}{\partial u^\sigma \partial u^\tau} \delta u^\tau = 0 \qquad i = 1,2, \text{ since } c_3 = 1.$$

The first of these, $\sigma = 1$, is, when expanded,

$$(5.2) \quad \begin{aligned} &(-p^1 \frac{v \sin u}{R \cos y} + q^1 \frac{v \cos u}{R} + r^1 f_u)\, dc_1 \\ &+ (-p^2 \frac{v \sin u}{R \cos y} + q^2 \frac{v \cos u}{R} + r^2 f_u) dc_2 \\ &+ (-p \frac{v \cos u}{R \cos y} - q \frac{v \sin u}{R} + r f_{uu})\, \delta u \\ &+ (-p \frac{\sin u}{R \cos y} + q \frac{\cos u}{R} + r f_{uv})\, \delta v \\ &+ r f_{uw}\, \delta w = 0; \end{aligned}$$

the second is, $\sigma = 2$,

$$(5.3) \quad \begin{aligned} &(p^1 \frac{\cos u}{R \cos y} + q^1 \frac{\sin u}{R} + r^1 f_v) dc_1 \\ &+ (p^2 \frac{\cos u}{R \cos y} + q^2 \frac{\sin u}{R} + r^2 f_v) dc_2 \\ &+ (-p \frac{\sin u}{R \cos y} + q \frac{\cos u}{R} + r f_{vu}) \delta u \\ &+ r f_{vv}\, \delta v + r f_{vw}\, \delta w = 0\,; \end{aligned}$$

and, for $\sigma = 3$,

$$(5.4) \quad f_{wu}\, \delta u + f_{wv}\, \delta v + f_{ww}\, \delta w = 0.$$

Comment: dc_1 is an ordinary differential, but δu is a function of t. If f is independent of u, several of the terms drop out. In any case, if the determinant

$$(5.5) \qquad \det(\vec{P}\cdot\vec{F}_{\sigma\tau})$$

is not zero, then we get $\delta u, \delta v, \delta w$ in terms of dc_1, dc_2 in equations of the form

$$\begin{aligned} \delta \bar{u} &= S^{11}dc_1 + S^{12}dc_2 \\ \delta v &= S^{21}dc_1 + S^{22}dc_2 \\ \delta w &= S^{31}dc_1 + S^{32}dc_2 \end{aligned}$$

where S^{ij} is a function of t. (If the determinant vanishes, the strong Legendre condition is not satisfied and the routine may break down.) When these are put into equation (3.14) for $\delta x(T)$, we get

$$\begin{aligned} (5.6) \qquad \delta x(T) &= \int_0^T \vec{P}^1\cdot\left(\frac{\partial \vec{F}}{\partial u} S^{11} + \frac{\partial \vec{F}}{\partial v} S^{21} + \frac{\partial \vec{F}}{\partial w} S^{31}\right)dt\ dc_1 \\ &+ \int_0^T \vec{P}^1\cdot\left(\frac{\partial \vec{F}}{\partial u} S^{12} + \frac{\partial \vec{F}}{\partial v} S^{22} + \frac{\partial \vec{F}}{\partial w} S^{32}\right)dt\ dc_2 \\ &= \sum_{\sigma,i} \int_0^T \vec{P}^1\cdot\vec{F}_\sigma S^{\sigma i}\ dt\ dc_i , \end{aligned}$$

with a similar expression, $\vec{P}^1 \rightarrow \vec{P}^2$ for $\delta y(T)$. These have the form

$$(5.7) \qquad dx^i(T) = \sum_1^2 a^{ij} dc_j$$

Let us calculate, along with the course, a fundamental set of solutions to the adjoint. We also solve equations (5.2), (5.3), (5.4) and substitute the result into (5.6) and its mate for

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$\delta y(T)$, to generate the integrals for a^{ij} in (5.7). Since a^{ij} is $\partial x^i(T)/\partial c_j$, we have the mathematical mechanism for a Newton-Raphson iteration. That is, to solve, we replace dx^i by $x_T^i - x^i(T)$, the desired minus the calculated value, and generate corrections for c_1, c_2:

$$(5.8) \qquad \begin{cases} x_T - x(T) = a^{11}\,\Delta c_1 + a^{12}\,\Delta c_2 \\ y_T - y(T) = a^{21}\,\Delta c_1 + a^{22}\,\Delta c_2 \end{cases}$$

This gives a new estimate of c_1, c_2.

The principal problem, as usual with a Newton-Raphson iteration is that of convergence. We may often force this as follows. Kee track of the error

$$(5.9) \qquad E = [x_T - x(T)]^2 + [y_T - y(T)]^2$$

If this does not diminish on successive iterations, divide the corrections obtained from (5.8) by, say, a factor of five. The Newton-Raphson iteration always moves the solution in the right direction to improve the solution, but it may overshoot badly. Once the values get close to the solution they tend to converge rapidly.

6. Control Variable on the Boundary, Corners.

In the above, it was generally assumed that the minimum value of H was attained for an interior value of w, neither the maximum nor minimum value. If the minimum value of H occurs for $w = W_{max}$, then equation (5.4) must be replaced by

$$\delta w = 0 \tag{6.1}$$

This may hold only in a certain intervals, or it may occur for the entire route; the variations are given by (5.2),(5.3) and (6.1) so long as w is on the boundary. So long as the variables are continuous, there is no particular problem; an extra test must be put in to determine whether the maximum is on the boundary, and when to leave it, etc.

However, it may be that in some cases the minimizing value of u,v,w is discontinuous. For example, if the listening devices are primarily on the surface initially, and in the later part of the route are deep, the submarine may well need to change from W_{max} to some lower value (a shallower route). A corner can also occur under other circumstances. The first condition for a corner is that there is a point t_1 of the route where two values of the control, say, u',v',w' and u'',v'',w'' both minimize the Hamiltonian H. That is, at this point t_1 of the route

$$H(\vec{X},\vec{U}',\vec{P},t) = H(\vec{X},\vec{U}'',\vec{P},t) = \min_{\vec{U}} \qquad (t = t_1) \tag{6.2}$$

Further, on the route,

$$\begin{aligned} H(\vec{X},\vec{U}',\vec{P},t) &> H(\vec{X},\vec{U}'',\vec{P},t) \quad \text{for} \quad t < t_1 \\ H(\vec{X},\vec{U}',\vec{P},t) &< H(\vec{X},\vec{U}'',\vec{P},t) \quad \text{for} \quad t_1 < t \end{aligned} \tag{6.3}$$

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in some neighborhood of t_1. If we try to solve for the variables by the minimum principle, we will get at least three sets of solutions, since there is usually a saddle point or a maximum associated with two relative minima of a continuous function.

This brings up the following difficulty. There is no general search procedure for finding the minimum of a function of several variables. It seems imperative that whoever programs the problem must have some feel for the likelihood of a corner, and have an idea where to look for the other minimum. A routine relying on gross computation will surely require excessive computation.

However, let us assume that we have a corner and we have located the two values $\vec{U}'$, $\vec{U}''$ for the control which yield equal values, in (6.2). Let us consider how a change in c_1, c_2 will change the Hamiltonian H. The change will be due to three terms. First, at a given point, H changes directly with c_1, c_2. Second changing c_1, c_2 will change $\vec{U}$, in accordance with (5.2),(5.3),(5.4), or if $w = W_{max}$, (5.4) is replaced by (6.1). Finally, if t_1 is changed, there is a term $\frac{dH}{dt}dt_1$. On an extremal, $\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \frac{\partial H}{\partial x^i}\dot{x}^j + \sum_i \frac{\partial H}{\partial p_i}\dot{p}_i + \sum_\sigma \frac{\partial H}{\partial u^\sigma}\dot{u}^\sigma = \frac{\partial H}{\partial t}$, when the original equations, and the Euler-Lagrange equations are used. Hence $dH(\vec{X},\vec{U}',\vec{P},t)$ can be put in the form

(6.4) $$dH' = \phi_1' dc_1 + \phi_2' dc_2 + \partial H'/\partial t \; dt_1,$$

the prime (') indicating that $\vec{U} = \vec{U}'$, and not a derivative. There is a similar

expression for dH'' at dt_1. At the corner the two values of H must be equal after the change in c_1, c_2, and t_1. Hence we get an equation which has the form

$$(6.5) \qquad \phi_1' dc_1 + \phi_2' dc_2 + \frac{\partial H'}{\partial t_1} dt_1 = \phi_1'' dc_1 + \phi_2'' dc_2 + \frac{\partial H''}{\partial t_1} dt_1 .$$

Hence, if we may assume, in light of (6.3) that

$$(\partial H''/\partial t)_{t_1} > (\partial H'/\partial t)_{t_1} ,$$

we can solve (6.5) for dt_1 in terms of dc_1, dc_2. This furnishes the value to put into the equations for $\delta x(T)$, $\delta y(T)$. In equation (5.6) there is adjoined a term

$$-[\vec{P}^1 \cdot \vec{F}]_{\vec{U}=\vec{U}'}^{\vec{U}=\vec{U}''} dt_1$$

for the corner and dt_1 is expressed in terms of dc_1, dc_2. When these terms are added to each equation in (5.7), they are unchanged in form.

This shows then how the routine is amended to handle corners.

We have not considered the way in which the likelihood of detection may vary with the heading angle. For any particular listening device, it may be a function of heading angle and if the signal strength varies greatly as the submarine turns, there may be corners as a consequence of this. These should be easy to check for, since the signal is probably symmetric with respect to the centerplane of the submarine. Fortunately, if a corner is missed by a small amount, the probability of detection is not increased much, since H and f must have the same value on both sides of the corner.

7. Comments.

The problem discussed so far is typical and indicates the principal points. My colleague Professor W. E. Bleick has made up Fortran programs for some of the simpler problems and has run them, without any particular difficulties arising.

Problems wherein the total time is free (that is, not specified), and wherein the final point is not given but only required to lie in a given region, are more tedious to program but are not essentially more difficult. The conditions on the route are not affected by the endcontitions except as the constants of integration in the adjoint system are changed.

It was initially hoped that a very general routine could be made up, one which would include all likely difficulties, with a subroutine adjoined to resolve each of them. It is not clear now whether this is feasible.

The principal difficulty lies in the nature of f and the way that it is generated and fitted. In the first place f will generally involve terms which are got from raw data, a collection of measurements. An examination of relevant data from Fleet Numerical Weather Facility suggests that there may be some problem in making up functions which have the required continuous derivatives with respect to the control variables. This part of the problem needs investigation. Probably most of the actual work of getting the routine to run and to yield good routes is in the subproblem of collecting the data and fitting it, so that the necessary accuracy is attained without excessive computational time.

If the data is too rough and irregular, it may be

necessary to go to steepest ascent methods and hope that there will be no corners. After the route is determined, we may check the corner condition, but this is time consuming. And if corners then need occur, the steepest ascent routine has an element of hit and miss in locating them.

A general treatment of corners is out of the question in programming, but that is not really needed here. The function $\bar{f}$ represents physical phenomena, and hence it will have some mathematical structure. If it is likely to have several relative minima, the people who work in the corresponding field will surely know this, and they must coordinate their work with the programmer so that he will know about when and where to look for these, and can organize the search program for them. This problem is not simple.

It seems necessary also that the programmers of these problems have some understanding of the theory; none of these problems turn out to be routine.

The optimum path generally has a nice property: If the submarine deviates from it by "small" amounts, the resulting route is "almost as good", since the route gives a stationary value to the probability and the first-order effects of changes are zero.

So far as the writer knows, there is no general treatment in the literature of the numerical solution of problems where the control lies on the boundary of the region of allowability, though some problems of this type have been solved by the writer and others. It was necessary in discussing solutions to examine this problem in detail and to examine the idea of normality and

attainability. Essentially these are questions concerning the conditions under which a point can be attained, and under which all neighboring endpoints can be attained, and of uniqueness. To answer these, it is necessary to study the rank of certain functional matrices, matrices whose elements are integrals (functionals) with arguments obtained from the route. It was planned to include this as an appendix to this report, but the theory is not yet complete and seems to merit a separate report, which is in preparation.

There are many interesting ramifications of this problem. For example, if we know the location of the listening devices accurately, and the water conditions exactly, then there are sometimes blind, or deaf, areas, regions wherein the submarine has very low likelihood of being detected, though in neighboring areas the likelihood is large, f being virtually discontinuous. The theory of solution of these problems is not understood. Similar problems arise when air defense has weapons or radar, say, with sharply defined limits.

The route may be changed as subsequent data is obtained. Indeed we may envision any route when followed by the submarine as a sequence of routes, with continual updating as subsequent data becomes available. If the data changes slowly, one or two iterations of the numerical routine will generate the necessary changes in the course.

The procedures given here can be readily extended to a problem such as that of sending a submarine to launch a missile, if data on the payoffs and risks is available. The payoff

balances the likelihood of damage or destruction to the target against the corresponding quantities for the submarine. The principal difficulty, in the writer's opinion, is the collection and fitting of data to describe the situation accurately.

For those interested in the earlier work on ship routing, it is summarized up fairly completely in the paper in Navigation [1] (the number in brackets refers to references listed at the end. The author and his colleagues have made a considerable comparison of the different methods; this is summarized in [2]. Some Fortran programs are available from Prof. W. E. Bleick. [3].

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[3] Bleick, W. E., Fortran Programs for Optimum Submarine Routes.

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13. ABSTRACT

A numerical routine is developed to determine on a digital computer a route that will enable a submarine to go from one specified point to another in a specified time, with minimum probability of being detected, when the likelihood of being detected is a known function of the position, heading, speed, depth, and time. Special routines are required to handle bounds on the control variable and corners.

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